SUBJECT: MATHEMATICS

PAPER: *NUMERICAL ANALYSIS*

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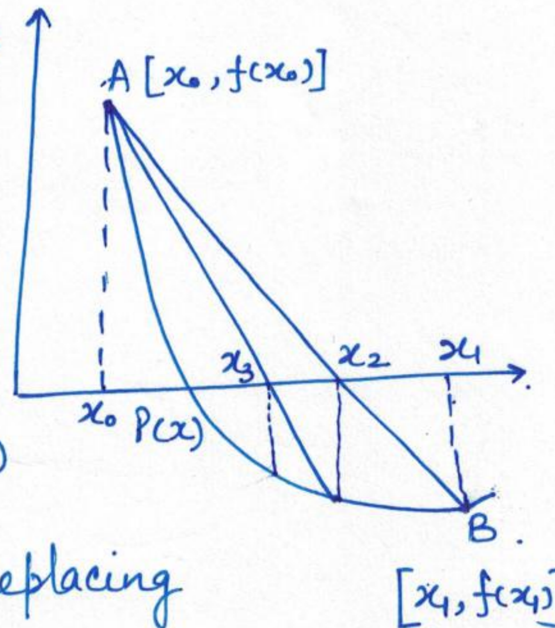
REGULA-FALSI METHOD (OR METHOD OF FALSE POSITION) :-

This is the oldest method for finding the real roots of the equation $f(x)=0$ and it is closely similar to the Bisection method.

In this method, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph of $y=f(x)$ crosses the x-axis between these two points, a root must lie between these points.

Consequently, $f(x_0)\,f(x_1) < 0$



Equation of the chord joining points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The method consists in replacing the curve AB by means of the chord AB and taking the points of intersection of the chord with x-axis as an approximation to the root.

This chord intersects the x-axis between the points where $y=0$. Thus we get

$$x = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\, f(x_0)$$

Hence the second approximation to the root is given by

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. Then replace the part of curve between the points $(x_0, f(x_0))$ and $(x_2, f(x_2))$ by the chord joining these points and this chord intersects the x-axis, where we get next approximation to the root and is therefore given by

$$x_3 = x_0 - \frac{(x_2 - x_0)}{f(x_2) - f(x_0)} f(x_0)$$

This procedure is repeated till the root is found to the desired accuracy.

<u>Remark</u> :- The order of convergence of Regula-falsi method is 1.618.

Example :- Find the real root of the equation $x^3-9x+1=0$ by Regula-falsi method.

Solution :- let $f(x) = x^3-9x+1=0$

then $f(1) = 1^3-9(1)+1 = -7$

$f(2) = 2^3-9(2)+1 = -9$

$f(3) = 3^3-9(3)+1 = 1$

since $f(2)$ and $f(3)$ are of opposite sign, therefore, the root lies between 2 and 3. .

Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = f(2) = -9$, $f(x_1) = f(3) = 1$

using method of false position,

$$x_2 = x_0 - \left\{\frac{x_1-x_0}{f(x_1)-f(x_0)}\right\} f(x_0)$$

$$= 2 - \left\{\frac{3-2}{1+9}\right\} \times (-9)$$

$$= 2.9$$

now $f(x_2) = f(2.9)$

$$= (2.9)^3 - 9(2.9)+1$$

$$= -0.711$$

Thus, the root lies between 2.9 and 3. Therefore, taking $x_0 = 2.9$, $x_1 = 3$, $f(x_0) = -0.711$, $f(x_1) = 1$.

Again using method of false position.

$$x_3 = x_0 - \left\{\frac{x_1-x_0}{f(x_1)-f(x_0)}\right\} f(x_0)$$

$$x_3 = 2.9 - \left\{\frac{3-2.9}{1+0.711}\right\}(-0.711)$$

$$= 2.9416$$

now $f(x_3) = f(2.9416)$

$$= (2.9416)^3 - 9(2.9416) + 1$$
$$= -0.0207$$

Thus the root lies between 2.9416 and 3. Therefore, taking $x_0 = 2.9416$, $x_1 = 3$, $f(x_0) = -0.0207$ $f(x_1) = 1$.

Then, we get

$$x_4 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= 2.9416 - \left\{\frac{3 - 2.9416}{1 + 0.0207}\right\}(-0.0207)$$

$$= 2.9428$$

$\therefore$ $f(x_4) = f(2.9428)$

$$= (2.9428)^3 - 9(2.9428) + 1$$
$$= -0.0003$$

Hence root lies between 2.9428 and 3. Therefore, taking $x_0 = 2.9428$, $x_1 = 3$, $f(x_0) = 0.0003$, $f(x_1) = 1$.

Again using method of false position,

$$x_5 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= 2.9428 - \left\{\frac{3 - 2.9428}{1 + 0.0003}\right\}(-0.0003)$$

$$= 2.942817$$

$\because$ x_4 and x_5 are same upto four decimal places hence the required root is 2.9428 correct to four decimal places.

Convergence of Regula-Falsi method :-

If $<x_n>$ be the sequence of approximations obtained from

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n) \quad \text{—} \; (1)$$

and α be the exact value of the root of equation $f(x) = 0$, then

Let $\qquad x_n = \alpha + e_n$

$$x_{n+1} = \alpha + e_{n+1}$$

where e_n, e_{n+1} being errors involved in n^{th} and $(n+1)^{th}$ approximations respectively. Clearly, $f(\alpha) = 0$.

Hence (1) gives,

$$\alpha + e_{n+1} = \alpha + e_n - \frac{(e_n - e_{n-1})}{f(\alpha+e_n) - f(\alpha+e_{n+1})} \cdot f(\alpha+e_n)$$

or, $$e_{n+1} = \frac{e_{n-1} f(\alpha+e_n) - e_n f(\alpha+e_{n-1})}{f(\alpha+e_n) - f(\alpha+e_{n-1})}$$

$$= \frac{e_{n-1}\left[f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{|\underline{2}} f''(\alpha) + \cdots\right] - e_n\left[f(\alpha) + e_{n-1} f'(\alpha) + \frac{e_{n-1}^2}{|\underline{2}} f''(\alpha) + \cdots\right]}{\left[f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{|\underline{2}} f''(\alpha) + \cdots\right] - \left[f(\alpha) + e_{n-1} f'(\alpha) + \frac{e_{n-1}^2}{|\underline{2}} f''(\alpha) + \cdots\right]}$$

$$\doteq \frac{(e_{n-1} - e_n) f(\alpha) + \frac{e_{n-1} e_n}{|\underline{2}} (e_n - e_{n-1}) f''(\alpha) + \cdots}{(e_n - e_{n-1}) f'(\alpha) + \frac{(e_n - e_{n-1})(e_n + e_{n-1})}{|\underline{2}} f''(\alpha) + \cdots}$$

$$= \frac{\frac{e_{n-1}e_n}{2} f''(\alpha) + ----}{f'(\alpha) + \left(\frac{e_n + e_{n-1}}{2}\right) f''(\alpha) + ---} \qquad [\because f(\alpha) = 0]$$

or $$e_{n+1} = \frac{e_n e_{n-1}}{\lfloor 2} \frac{f''(\alpha)}{f'(\alpha)} \quad ---- (2)$$

(neglecting high powers of e_n, e_{n-1})

Let $e_{n+1} = c\, e_n^k$, where c is a constant and $k > 0$.

$\therefore \quad e_n = c\, e_{n-1}^k$

or $\quad e_{n-1} = c^{-1/k} e_n^{1/k}$

$\therefore$ from eqn (2),

$$c\, e_n^k = \frac{e_n \cdot c^{-1/k} e_n^{1/k}}{\lfloor 2} \frac{f''(\alpha)}{f'(\alpha)}$$

$$= \frac{c^{-1/k}}{\lfloor 2} e_n^{1+\frac{1}{k}} \frac{f''(\alpha)}{f'(\alpha)}$$

comparing both sides, we get

$$k = 1 + \frac{1}{k} \quad \text{and} \quad c = \frac{c^{-1/k}}{\lfloor 2} \frac{f''(\alpha)}{f'(\alpha)}$$

Now, $k = 1 + \frac{1}{k} \Rightarrow k^2 - k - 1 = 0 \Rightarrow k = 1.618$

Also $$c = \frac{c^{-1/k}}{\lfloor 2} \frac{f''(\alpha)}{f'(\alpha)}$$

$$c^{1+\frac{1}{k}} = c^{1.618} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$$

or $$c = \left[\frac{f''(\alpha)}{2 f'(\alpha)}\right]^{0.618}$$

This gives rate of convergence and $k = 1.618$ gives the order of convergence.

SECANT METHOD :-

This method is quite similar to that of Regula-falsi method except for the condition $f(x_0) \cdot f(x_1) < 0$. Here the graph of the function $y = f(x)$ in the neighbourhood of the root is approximated by a secant line or chord. Taking x_0, x_1 as the initial limits of the interval, then the equation of the chord joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is given by

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_1)$$

This secant line cuts the x-axis ie $y = 0$, then the abscissa of the point (or first approximation) is given by

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1)$$

Again, the formula for successive approximation in general is

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] f(x_n)$$

In case at any stage $f(x_n) = f(x_{n-1})$, this method will fails and shows that it does not converges necessarily. If the secant method converges, its rate of convergence is faster than that of the method of false position.

Example :- A real root of the equation

$$f(x) = x^3 - 5x + 1 = 0 .$$

lies in the interval (0,1). Perform four iterations of the secant method.

Solution :- we have $x_0 = 0$, $x_1 = 1$, $f(x_0) = 1$, $f(x_1) = -3$

By Secant method, first approximation is

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1)$$

$$= 1 - \left[\frac{1-0}{-3-1}\right](-3)$$

$$= 0.25$$

$$\therefore f(x_2) = -0.234375$$

Second approximation is

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] f(x_2)$$

$$= 0.25 - \left[\frac{0.25 - 1}{-0.234375 + 3}\right](-0.234375)$$

$$= 0.186441$$

$$\therefore f(x_3) = 0.074276$$

Third approximation is

$$x_4 = x_3 - \left[\frac{x_3 - x_2}{f(x_3) - f(x_2)}\right] f(x_3)$$

$$= 0.186441 - \left[\frac{0.186441 - 0.25}{0.074276 + 0.234375}\right](0.074276)$$

$$= 0.201736$$

$$\therefore f(x_4) = -0.000470$$

Fourth approximation is

$$x_5 = x_4 - \left[\frac{x_4 - x_3}{f(x_4) - f(x_3)}\right] f(x_4)$$

$$= 0.201640$$

Assignment :- **Q1**. Find the real root of the equation $x^3-x^2-2=0$ by Regula-falsi method.

Ans :- $\boxed{1.695}$

Q2. Find the real root of the equation $x^2-2x-1=0$ lies between 1 and 3; correct to three decimal places.

Ans :- $\boxed{2.4141}$

Q3. Determine a real root of the equation $xe^{x}-3=0$, using Regula-falsi method correct to three decimal places.

Ans :- $\boxed{1.0498}$

Q4. Solve $x^3-5x+3=0$ by using Regula-falsi method.

Ans :- $\boxed{1.7963}$

Q5. Find the rate of convergence of Secant method.

Ans :- $\boxed{1.62}$

Q6 : Determine the root of the equation $f(x)=\cos x - xe^{x}=$ using the secant method upto four decimal places.

Ans :- $\boxed{0.5177}$

Q7. Compute root of the equation $x^2 e^{-x/2}=1$ in the interval $[0,2]$ using secant method. The root should be correct to three decimal places.

Ans :- $\boxed{1.4292}$